Color Superconductivity in High Density Effective Theory

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Abstract

In this talk, I discuss the recent development in color superconductivity in terms of effective field theory. By investigating the Cooper pair gap equations at high density, we see that the effective theory simplifies the gap analysis very much, especially in finding the ground state, the precise form of the gap, and the critical temperature. Furthermore, the effective theory enables us to estimate the critical density for color superconductivity, which is found to be around 230 MeV in the hard-dense-loop approximation. Finally, I briefly mention the low-lying spectra of color superconductor at high density.

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1 Introduction

At high density, quarks in dense matter interact weakly with each other and form a Fermi sea, due to asymptotic freedom. When the energy is much less than the quark chemical potential ($E \ll \mu$), only the quarks near the Fermi surface are relevant. The dynamics of quarks near the Fermi surface is effectively one-dimensional, since excitations along the Fermi surface do not cost any energy. The momentum perpendicular to the Fermi momentum just labels the degeneracy, similarly to the perpendicular momentum of charged particle under external magnetic field. This dimensional reduction due to the presence of Fermi surface makes possible for quarks to form a Cooper pair for any arbitrary weak attraction, since the critical coupling for the condensation in (1+1) dimensions is zero, known as the Cooper theorem in condensed matter.

While, in the BCS theory, such attractive force for electron Cooper pair is provided by phonons, for dense quark matter, where phonons are absent, the gluon exchange interaction provides the attraction, as one-gluon exchange interaction is attractive in the color anti-triplet channel. One therefore expects that color anti-triplet Cooper pairs will form and quark matter is color superconducting, which is indeed shown more than 20 years ago [1, 2, 3].

Recent development in color superconductivity, started from 1998, was spurred by recent two seminal works. The first one is by Alford, Rajagopal, and Wilczek [4], who convincingly argued that for three massless flavors, the ground state of quark matter is a so-called color-flavor locking (CFL) phase, in which the Cooper pair takes the following form, neglecting the small sextet component,

$$\left\langle \psi^a_{L\alpha}(\vec{p})\psi^b_{L\beta}(-\vec{p})\right\rangle = -\left\langle \psi^a_{R\alpha}(\vec{p})\psi^b_{R\beta}(-\vec{p})\right\rangle = \epsilon^{abI}\epsilon_{\alpha\beta I}K(p_F), \tag{1}$$

where a, b $(= 1, 2, 3)$ denote the color indices and α, β $(= 1, 2, 3)$ denote the flavor indices.

The interesting feature of the CFL phase is that chiral symmetry is broken and the excitations in CFL phase have integral multiplet of electron charge. Though the usual quark-antiquark condensate is absent at high density, at least at the leading order, the chiral symmetry is spontaneously broken in the CFL phase. The flavor indices of Cooper pairs are locked to their color indices so that the unbroken symmetry that leaves the Cooper pair condensate invariant is the simultaneous rotation in the flavor and color space, breaking both color and chiral symmetry down to their diagonal subgroup,

$$SU(3)_c \times SU(3)_L \times SU(3)_R \to SU(3)_{c+L+R}. \tag{2}$$

The second work is done by Son [5], who showed that the Cooper pair gap in high density quark matter is very different from the usual BCS gap, due to the long range (color) magnetic

interaction among quarks. By the renormalization group (RG) analysis, aided by the analysis of the Eliashberg equation, he found the Cooper pair gap depends on the coupling as,

$$\Delta \sim \frac{\mu}{g_s^5} \exp\left(-\frac{3\pi^2}{\sqrt{2}g_s}\right), \tag{3}$$

which was confirmed by more careful analysis [6, 7, 8, 9, 10, 11].

In this talk, I will derive the above results in terms of a high density effective theory derived in [6, 7]. I will also calculate the critical temperature and the critical density and mention the mass of low-lying excitations in the CFL phase.

2 High density effective theory

QCD at high density has two distinct scales; one is an extrinsic scale, μ, the quark chemical potential, and the other is the intrinsic scale, $\Lambda_{\rm QCD}$. If the density is high enough, two scales are well separated, $\mu \gg \Lambda_{\rm QCD}$. To study a low-energy physics below a scale Λ, an effective theory approach, where heavy modes ($\omega > \Lambda$) are separated from light modes ($\omega < \Lambda$) systematically, has been quite powerful.

Since we are interested in a cold dense matter where the relevant excitations are quasi-quarks near the Fermi surface, it will be useful to construct an effective theory that deals only with those relevant degrees of freedom [6, 7]. A dense matter with a fixed quark number is described by the QCD Lagrangian density with a chemical potential μ,

$$\mathcal{L}_{\rm QCD} = \bar{\psi} i \not{D} \psi - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \mu \bar{\psi}\gamma_0\psi, \tag{4}$$

where the covariant derivative $D_\mu = \partial_\mu + i g_s A^a_\mu T^a$ and we neglect the mass of quarks for simplicity.

At energy just below μ, we may decompose the quark momentum as

$$p^\mu = \mu v^\mu + l^\mu, \quad |l^\mu| < \mu, \tag{5}$$

where $\vec{v}_F$ is a Fermi velocity and $v^\mu = (0, \vec{v}_F)$. We expand the quark propagator in powers of $1/\mu$:

$$S_F(p) = \frac{i}{(1+i\epsilon)p^0\gamma^0 - \vec{p}\cdot\vec{\gamma} + \mu\gamma^0} = P_+ \frac{i\gamma^0}{l\cdot V + i\epsilon l^0} + P_- \frac{i\gamma^0}{2\mu} + \cdots, \tag{6}$$

where $V^\mu = (1, \vec{v}_F)$ and the ellipsis denote higher order terms in $1/\mu$ expansion. In the second line of Eq. (6) we have introduced projection operators

$$P_\pm = \frac{1 \pm \vec{\alpha}\cdot\vec{v}_F}{2}, \tag{7}$$

where $\vec{\alpha} = \gamma^0 \vec{\gamma}$. The projection operators P_+ and P_- project out the states near the Fermi surface and the states in the Dirac sea, respectively. We see that the propagating modes are the states near the Fermi surface.

Using the techniques developed in heavy quark effective theory [12], we Fourier-decompose the quark field as

$$\psi(x) = \sum_{\vec{v}_F} e^{i\mu \vec{v}_F \cdot \vec{x}} \psi(\vec{v}_F, x) \tag{8}$$

where

$$\psi(\vec{v}_F, x) = \int_{|l^\mu|<\mu} \frac{d^4 l}{(2\pi)^4} \psi(\vec{v}_F, l) e^{-il\cdot x} \equiv \psi_+(\vec{v}_F, x) + \psi_-(\vec{v}_F, x) \tag{9}$$

with $\psi_\pm(\vec{v}_F, x) = P_\pm \psi(\vec{v}_F, x)$. The low energy effective Lagrangian that consists of the light degrees of freedom (gluons and ψ_+) is obtained by matching all one-light-particle irreducible amplitudes in QCD with the vertex functions in the effective theory. As shown in [6, 7], in the effective theory the quark propagator becomes

$$S_F(\vec{v}_F; l) = \frac{1 + \vec{\alpha} \cdot \vec{v}_F}{2} \frac{i\gamma^0}{l \cdot V + i\epsilon l^0}, \tag{10}$$

and in addition to the quark-gluon minimal coupling $-i\gamma^0 V^\mu g_s$ there is marginal four-Fermi interaction for quarks with opposite Fermi velocities,

$$\begin{aligned} \mathcal{L}^1_{4\mathrm{f}} &= \frac{g^S_{us;tv}}{2\mu^2} \left[\psi^\dagger_{Lt}(\vec{v}_F, x)\psi_{Ls}(\vec{v}_F, x)\psi^\dagger_{Lv}(-\vec{v}_F, x)\psi_{Lu}(-\vec{v}_F, x) + (L \leftrightarrow R) \right] \\ &+ \frac{g^P_{us;tv}}{2\mu^2} \left[\psi^\dagger_{Lt}(\vec{v}_F, x)\psi_{Ls}(\vec{v}_F, x)\psi^\dagger_{Rv}(-\vec{v}_F, x)\psi_{Ru}(-\vec{v}_F, x) + (L \leftrightarrow R) \right]. \end{aligned} \tag{11}$$

To summarize, the high density effective theory has several interesting features: (1) In the leading order, only γ^0 enters in the Dirac matrices. (2) Anti-quarks are systematically decoupled. (3) There appear marginal four-quark operators naturally. (4) It offers a systematic high-density expansion.

3 Cooper pair gap

To describe the Cooper-pair gap equation, we introduce a 8-component field, following the Nambu-Gorkov formalism, $\Psi(\vec{v}_F, x) \equiv (\psi(\vec{v}_F, x), \psi_c(\vec{v}_F, x))^T$, where we reverted the notation ψ for ψ_+ and introduced the charge conjugate field $\psi_c(\vec{v}_F, x) = C\bar{\psi}^T(-\vec{v}_F, x)$. The charge conjugation matrix, C, satisfies $C^{-1}\gamma_\mu C = -\gamma^T_\mu$. The inverse propagator for the Nambu-Gorkov field is

$$S^{-1}(\vec{v}_F, l) = -i\gamma_0 \begin{pmatrix} l \cdot V & -\Delta^\dagger(l_\parallel) \\ -\Delta(l_\parallel) & l \cdot \bar{V} \end{pmatrix}, \tag{12}$$

where $\bar{V}^\mu = (1, -\vec{v}_F)$ and Δ is the Cooper-pair gap.

The effective action for the fermion two-point function S is given as

$$\Gamma = -\mathrm{Tr}\ln S^{-1} + \mathrm{Tr}\left(S^{-1} - S_0^{-1}\right) S + (\text{2PI diagrams})\,, \tag{13}$$

where the 2PI diagrams are two-particle irreducible vacuum diagrams. For the gluon propagator, we use an in-medium propagator, which is in the hard dense loop (HDL) approximation given as

$$iD_{\mu\nu}(k) = \frac{P^T_{\mu\nu}}{k^2 - G} + \frac{P^L_{\mu\nu}}{k^2 - F} - \xi\frac{k_\mu k_\nu}{k^4}, \tag{14}$$

where ξ is the gauge parameter and the projectors are defined by

$$P^T_{ij} = \delta_{ij} - \frac{k_i k_j}{|\vec{k}|^2}, \quad P^T_{00} = 0 = P^T_{0i} \tag{15}$$

$$P^L_{\mu\nu} = -g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2} - P^T_{\mu\nu}. \tag{16}$$

The medium effect is incorporated in F and G, which becomes in the weak coupling limit ($|k_0| \ll \left|\vec{k}\right|$)

$$F(k_0, \vec{k}) \simeq M^2, \quad G(k_0, \vec{k}) \simeq \frac{\pi}{4} M^2 \frac{k_0}{|\vec{k}|}, \tag{17}$$

where $M = \sqrt{N_f/2}\; g_s\mu/\pi$, the Debye mass. The gap equations, obtained by extremizing the effective action, $0 = \delta\Gamma/\delta S$, are given in Euclidean space as

$$\begin{aligned}\Delta(p_\parallel) = \int \frac{d^4q}{(2\pi)^4} \Bigg[-\frac{2}{3} g_s^2 \Bigg\{ & \frac{V \cdot P^T \cdot \bar{V}}{(p-q)_\parallel^2 + \vec{q}_\perp^{\,2} + \frac{\pi}{4} M^2 |p_0 - q_0| / |\vec{p} - \vec{q}|} \\ & - \frac{1}{(p-q)_\parallel^2 + \vec{q}_\perp^{\,2} + M^2} - \xi\frac{(p-q)_\parallel^2}{(p-q)^4} \Bigg\} + \frac{g_{\bar{3}}}{\mu^2} \Bigg] \frac{\Delta(q_\parallel)}{q_\parallel^2 + \Delta^2(q_\parallel)},\end{aligned} \tag{18}$$

where $g_{\bar{3}}$ is a four-Fermi coupling. Since the gluon coupling is vectorial, the gluon exchange interaction in the gap equation does not distinguish the handedness of quarks and thus it will generate same condensates regardless of handedness; $|\langle\psi_L\psi_L\rangle| = |\langle\psi_R\psi_R\rangle| = |\langle\psi_L\psi_R\rangle|$, suppressing other quantum numbers. But, the four-Fermi interaction in the effective Lagrangian, Eq. (11), lifts the degeneracy, since the gap in LL or RR channel will be bigger than the one in LR channel due to the difference in the four-Fermi couplings, $g^S > g^P$. The LL or RR condensate is energetically more preferred than the LR condensate. We also note that since in the effective theory the gluons are blind not only to flavors but also to the Dirac indices of quarks, the diquark Cooper-pair can be written as color anti-triplet.

Since quarks are anti-commuting, the only possible way to form diquark (S-wave) condensate is either in spin-singlet or in spin-triplet:

$$\left\langle \psi_{L}{}^{a}_{i\alpha}(\vec{v}_F, x)\psi_{L}{}^{b}_{j\beta}(-\vec{v}_F, x)\right\rangle = -\left\langle \psi_{R}{}^{a}_{i\alpha}(\vec{v}_F, x)\psi_{R}{}^{b}_{j\beta}(-\vec{v}_F, x)\right\rangle \tag{19}$$

$$= \epsilon_{ij}\epsilon^{abc}K_{[\alpha\beta]c}(p_F) + \delta_{ij}\epsilon^{abc}K_{\{\alpha\beta\}c}(p_F), \tag{20}$$

where $a, b, c = 1, 2, 3$ are color indices, $\alpha, \beta, \gamma = u, d, s, \cdots, N_f$ flavor indices, and $i, j = 1, 2$ spinor indices. Indices in the bracket and in the curled bracket are anti-symmetrized and symmetrized, respectively. But, the spin-one component of the gap, $K_{\{\alpha\beta\}c}$, vanishes algebraically, since $\psi(\vec{v}_F, x) = 1/2\,(1 + \vec{\alpha}\cdot\vec{v}_F)\,\psi(\vec{v}_F, x)$ and $(1 + \vec{\alpha}\cdot\vec{v}_F)_{il}(1 - \vec{\alpha}\cdot\vec{v}_F)_{lj} = 0$.

When $N_f = 3$, the spin-zero component of the condensate becomes (flavor) anti-triplet,

$$K_{[\alpha\beta]c}(p_F) = \epsilon_{\alpha\beta\gamma}K_c^{\gamma}(p_F). \tag{21}$$

Using the global color and flavor symmetry, one can always diagonalize the spin-zero condensate as $K_c^{\gamma} = \delta_c^{\gamma}K_{\gamma}$. To determine the parameters, K_u, K_d, and K_s, we need to minimize the vacuum energy for the condensate. The vacuum energy is given as in the leading HDL approximation

$$V(\Delta) \simeq \frac{\mu^2}{4\pi}\sum_{i=1}^{9}\int\frac{d^2l_{\parallel}}{(2\pi)^2}\left[\ln\left(\frac{l_{\parallel}^2}{l_{\parallel}^2 + \Delta_i^2(l_{\parallel})}\right) + \frac{1}{2}\cdot\frac{\Delta_i^2(l_{\parallel})}{l_{\parallel}^2 + \Delta_i^2(l_{\parallel})}\right], \tag{22}$$

where Δ_i's are the eigenvalues of color anti-symmetric and flavor anti-symmetric 9×9 gap, $\Delta_{\alpha\beta}^{ab}$.

Approximating Δ_i to be constant, one can easily perform the momentum integration in (22) to get

$$V(\Delta) \simeq -0.43\frac{\mu^2}{4\pi^2}\sum_i\left|\Delta_i(0)\right|^2. \tag{23}$$

Were Δ_i independent of each other, the global minimum should occur at $\Delta_i(0) = \text{const.}$ for all $i = 1, \cdots, 9$. But, due to the global color and flavor symmetry, only three of them are independent. Similarly to the condensate, the gap can be also diagonalized by the color and flavor symmetry as

$$\Delta_{ab}^{\alpha\beta} = \epsilon_{\alpha\beta\gamma}\epsilon^{abc}\Delta_{\gamma}\delta_c^{\gamma}. \tag{24}$$

Without loss of generality, we can take $|\Delta_u| \geq |\Delta_d| \geq |\Delta_s|$. Let $\Delta_d/\Delta_u = x$ and $\Delta_s/\Delta_u = y$. Then, the vacuum energy becomes

$$V(\Delta) \simeq -0.43\frac{\mu^2}{4\pi^2}\left|\Delta_u\right|^2 f(x, y), \tag{25}$$

where $f(x,y)$ is a complicate function of $-1 \le x, y \le 1$ that has a maximum at $x = 1 = y$, $f(x,y) \le 13.4$. Therefore, the vacuum energy has a global minimum when $\Delta_u = \Delta_d = \Delta_s$, or in terms of the eigenvalues of the gap

$$\Delta_i = \Delta_u \cdot (1, 1, 1, -1, 1, -1, 1, -1, -2). \tag{26}$$

Now, we analyze the SD gap equation Eq. (18) to see if it admits a nontrivial solution. Since the color-flavor locking gap is preferred if it exists, we may write the gap as

$$\Delta^{ab}_{\alpha\beta} = \epsilon^{abI}\epsilon_{\alpha\beta I}\Delta. \tag{27}$$

We first note that the main contribution to the integration comes from the loop momenta in the region $q_{\parallel}^2 \sim \Delta^2$ and $|\vec{q}_{\perp}| \sim M^{2/3}\Delta^{1/3}$. Therefore, we find that the leading contribution is by the first term due to the Landau-damped magnetic gluons. For this momentum range, we can take $|\vec{p} - \vec{q}| \sim |\vec{q}_{\perp}|$ and

$$V \cdot P^T \cdot \bar{V} = -v_F^i v_F^j \left(\delta_{ij} - \hat{k}_i \hat{k}_j\right) = -1 + O\left(\frac{\Delta^{4/3}}{M^{4/3}}\right). \tag{28}$$

We also note that the term due to the four-Fermi operator is negligible, since $g_{\bar{3}} \sim g_s^4$ at the matching scale μ.

Neglecting $(p-q)_{\parallel}^2$ in the denominator to integrate over $\vec{q}_{\perp}$, we get

$$\Delta(p_{\parallel}) = \frac{g_s^2}{9\pi}\int \frac{d^2q_{\parallel}}{(2\pi)^2}\frac{\Delta(q_{\parallel})}{q_{\parallel}^2 + \Delta^2}\left[\ln\left(\frac{\mu^3}{\frac{\pi}{4}M^2|p_0 - q_0|}\right) + \frac{3}{2}\ln\left(\frac{\mu^2}{M^2}\right) + \frac{3}{2}\xi\right]. \tag{29}$$

We see that in this approximation $\Delta(p_{\parallel}) \simeq \Delta(p_0)$. Then, we can integrate over $\vec{v}_F \cdot \vec{q}$ to get

$$\Delta(p_0) \;=\; \frac{g_s^2}{36\pi^2}\int_{-\mu}^{\mu} dq_0 \frac{\Delta(q_0)}{\sqrt{q_0^2 + \Delta^2}} \ln\left(\frac{\bar{\Lambda}}{|p_0 - q_0|}\right) \tag{30}$$

where $\bar{\Lambda} = 4\mu/\pi \cdot (\mu/M)^5 e^{3/2\xi}$. If we take $\Delta \simeq \Delta(0)$ for a rough estimate of the gap,

$$1 = \frac{g_s^2}{36\pi^2}\left[\ln\left(\frac{\bar{\Lambda}}{\Delta}\right)\right]^2 \quad \text{or} \quad \Delta \simeq \bar{\Lambda}\exp\left(-\frac{6\pi}{g_s}\right). \tag{31}$$

To take into account the energy dependence of the gap, we convert the Schwinger-Dyson equation (30) into a differential equation, approximating the kernel as

$$\ln|p_0 - q_0| \simeq \ln\left[\max.\left(|p_0|, |q_0|\right)\right], \tag{32}$$

to get

$$p\Delta''(p) + \Delta'(p) + \frac{2\alpha_s}{9\pi}\frac{\Delta(p)}{\sqrt{p^2 + \Delta^2}} = 0, \tag{33}$$

with boundary conditions $p\Delta' = 0$ at $p = \Delta$ and $\Delta = 0$ at $p = \bar{\Lambda}$, where $p \equiv p_0$. When $p \ll \Delta(p)$, the equation becomes

$$p\Delta'' + \Delta' + \frac{r^2}{4}\frac{\Delta(p)}{|\Delta|} = 0, \tag{34}$$

where $r^2 = 2g_s^2/(9\pi^2)$ and $|\Delta|$ is the gap at $p = 0$. We find $\Delta(p) = |\Delta| J_0\left(r\sqrt{p/|\Delta|}\right)$ for $p \ll \|\Delta|$. When $p \gg \Delta$, the differential equation (33) becomes

$$p\Delta'' + \Delta' + \frac{r^2}{4}\frac{\Delta}{p} = 0, \tag{35}$$

whose solution is $\Delta(p) = B\sin\left[(r/2)\ln\bar{\Lambda}/p\right]$. By matching two solutions at the boundary $p = |\Delta|$ we get

$$B \simeq |\Delta| \quad \text{and} \quad |\Delta| = \bar{\Lambda}e^{-\pi/r}. \tag{36}$$

The gap is therefore given as at the leading order in the weak coupling expansion

$$|\Delta| = c\cdot\frac{\mu}{g_s^5}\exp\left(-\frac{3\pi^2}{\sqrt{2}g_s}\right), \tag{37}$$

where $c = 2^7\pi^4 N_f^{-5/2}e^{3\xi/2+1}$. This agrees with the RG analysis done by Son [5] (see also [11]) and also with the Schwinger-Dyson approach in full QCD [8, 9, 10]. The $1/g_s$ behavior of the exponent of the gap at high density is due to the double logarithmic divergence in the gap equation (18), similarly to the case of chiral symmetry breaking under external magnetic fields [13, 14].

4 Critical density and temperature

In this section we calculate the critical density and temperature. First, we add the $1/\mu$ corrections to the gap equation Eq. (18) to see how the formation of Cooper pair changes when the density decreases. The leading $1/\mu$ corrections to the quark-gluon interactions are

$$\mathcal{L}_1 = -\frac{1}{2\mu}\sum_{\vec{v}_F}\psi^\dagger(\vec{v}_F,x)\left(\gamma_\perp\cdot D\right)^2\psi(\vec{v}_F,x) = -\sum_{\vec{v}_F}\left[\psi^\dagger\frac{D_\perp^2}{2\mu}\psi + g_s\psi^\dagger\frac{\sigma_{\mu\nu}F^{\mu\nu}}{4\mu}\psi\right]. \tag{38}$$

In the leading order in the HDL approximation, the loop correction to the vertex is neglected and the quark-gluon vertex is shifted by the $1/\mu$ correction as

$$-ig_s v_F^i \mapsto -ig_s v_F^i - ig_s\frac{l_\perp^i}{\mu}, \tag{39}$$

where l_i is the momentum carried away from quarks by gluons. We note that since the $1/\mu$ correction to the quark-gluon vertex does not depend on the Fermi velocity of the quark, it

generates a repulsion for quark pairs. For a constant gap approximation, $\Delta(p_\parallel) \simeq \Delta$, the gap equation becomes in the leading order, as $p \to 0$,

$$1 = \frac{g_s^2}{9\pi}\int \frac{\mathrm{d}^2 l_\parallel}{(2\pi)^2}\left[\ln\left(\frac{\bar{\Lambda}}{|l_0|}\right) - \frac{3}{2}\right]\frac{1}{l_\parallel^2 + \Delta^2} = \frac{g_s^2}{36\pi^2}\ln\left(\frac{\bar{\Lambda}}{\Delta}\right)\left[\ln\left(\frac{\bar{\Lambda}}{\Delta}\right) - 3\right]. \tag{40}$$

When $\mu < \mu_c \simeq e^3\Delta$, the gap due to the long-range color magnetic interaction disappears. Since the phase transition for color superconducting phase is believed to be of first order [15, 16], we may assume that the gap has the same dependence on the chemical potential μ as the leading order. Then, the critical density for the color superconducting phase transition is given by

$$\mu_c = e^3 \mu_c \exp\left[-\frac{3\pi^2}{\sqrt{2} g_s(\mu_c)}\right]. \tag{41}$$

Therefore, if the strong interaction coupling is too strong at the scale of the chemical potential, the gap does not form. To form the Cooper pair gap, the strong coupling at the scale of the chemical potential has to be smaller than $g_s(\mu_c) = \pi^2/\sqrt{2}$. By using the one-loop β function for three light flavors, $\beta(g_s) = -9/(16\pi^2)g_s^3$, and the experimental value for the strong coupling constant, $\alpha_s(1.73\mathrm{GeV}) = 0.32^{+0.031}_{-0.053}(\mathrm{exp}) \pm 0.016(\mathrm{theo})$ [17], we get $0.13\mathrm{GeV} < \mu_c < 0.31\mathrm{GeV}$, which is about the same order as the one estimated by the instanton induced four-Fermi interaction [16, 18] or by general effective four-Fermi interactions [15]. But, this should be taken as an order of magnitude, since for such a small chemical potential the higher order terms in $1/\mu$ expansion, which we have neglected, are as important as the leading term.

We now consider the temperature effect on the gap, which is quite important to understand the heavy ion collision or the final stage of the evolution of giant stars. The super dense and hot quark matter will go through a phase transition as it cools down by emitting weakly interacting particles like neutrinos.

At finite temperature, T, the gap equation (18) becomes, following the imaginary-time formalism developed by Matsubara,

$$\Delta(\omega_{n'}) = \frac{g_s^2}{9\pi}T\sum_{n=-\infty}^{+\infty}\int\frac{\mathrm{d}q}{2\pi}\frac{\Delta(\omega_n)}{\omega_n^2 + \Delta^2(\omega_n) + q^2}\ln\left(\frac{\bar{\Lambda}}{|\omega_{n'} - \omega_n|}\right), \tag{42}$$

where $\omega_n = \pi T(2n+1)$ and $q \equiv \vec{v}_F\cdot\vec{q}$. We now use the constant (but temperature-dependent) gap approximation, $\Delta(\omega_n) \simeq \Delta(T)$ for all n. Taking $n' = 0$ and converting the logarithm into integration, we get

$$\Delta(T) = \frac{g_s^2}{18\pi}T\sum_{n=-\infty}^{+\infty}\int\frac{\mathrm{d}q}{2\pi}\int_0^{\bar{\Lambda}^2}\mathrm{d}x\frac{\Delta(T)}{\omega_n^2 + \Delta^2(T) + q^2}\cdot\frac{1}{x + (\omega_n - \omega_0)^2}. \tag{43}$$

Using the contour integral, one can in fact sum up over all n to get

$$1=\frac{g_s^2T}{36\pi^2}\int \mathrm{d}q\int_0^{\bar{\Lambda}^2}\mathrm{d}x\frac{1}{2\pi i}\oint_C\frac{\mathrm{d}\omega}{1+e^{-\omega/T}}\cdot\frac{1}{(\omega^2-q^2-\Delta^2)\left[(\omega_n-i\omega_0)^2+x\right]}. \tag{44}$$

Since the gap vanishes at the critical temperature, $\Delta(T_C)=0$, after performing the contour integration in Eq. (44), we get

$$\begin{aligned}1 &= \frac{g_s^2}{36\pi^2}\int \mathrm{d}q\int_0^{\bar{\Lambda}^2}\mathrm{d}x\left\{\frac{(\pi T_C)^2+x-q^2}{\left[(\pi T_C)^2+x-q^2\right]^2+(2\pi T_C q)^2}\cdot\frac{\tanh\left[q/(2T_C)\right]}{2q}\right.\\ &\quad\left.+\frac{(\pi T_C)^2+q^2-x}{\left[(\pi T_C)^2+q^2-x\right]^2+(2\pi T_C)^2x}\cdot\frac{\coth\left[\sqrt{x}/(2T_C)\right]}{\sqrt{2}}\right\}.\end{aligned} \tag{45}$$

At high density $\bar{\Lambda}\gg T_C$, the second term in the integral in Eq. (45) is negligible, compared to the first term, and integrating over x, we get

$$\begin{aligned}1 &= \frac{g_s^2}{36\pi^2}\int_0^{\lambda_c}\mathrm{d}y\frac{\tanh y}{y}\left[\ln\left(\frac{\lambda_c^2}{(\pi/2)^2+y^2}\right)+O\left(\frac{y^2}{\lambda_c^2}\right)\right]\\ &= \frac{g_s^2}{36\pi^2}\left[(\ln\lambda_c)^2+2A\ln\lambda_c+\text{const.}\right]\end{aligned}$$

where we have introduced $y\equiv q/(2T_C)$ and $\lambda_c\equiv\bar{\Lambda}/(2T_C)$ and A is given as

$$A=\int_0^1\mathrm{d}y\frac{\tanh y}{y}+\int_1^\infty\mathrm{d}y\frac{\tanh y-1}{y}=\ln\left(\frac{4}{\pi}\right)+\gamma. \tag{46}$$

Therefore, we find, taking the Euler-Mascheroni constant $\gamma\simeq 0.577$,

$$T_C=\frac{e^A}{2}\Delta\simeq 1.13\ \Delta, \tag{47}$$

which shows that the ratio between the critical temperature and the Cooper-pair gap in color superconductivity is same as the BCS value, $e^\gamma/\pi\simeq 0.57$ [10, 19].

5 More on CFL

As pointed out by Schäfer and Wilczek [21], the low-lying particle spectrum of the CFL phase at high density resembles that of low density hadron phase. Both phases have pions and kaons, arising from the chiral symmetry breaking. The baryons and mesons at high density have integral multiplet of the electron charge, the charge corresponding to the unbroken $U(1)$ gauge symmetry at high density. Since the diquark condensate provides additional baryon number $B=2/3$, quarks in color superconductor have baryon number $B=1$.

To describe the dynamics of pions and kaons, the chiral Lagrangian for the CFL phase at high density has been constructed [22, 23] and it is shown in [22] that quarks in the CFL

phase is realized as a topological soliton, called superqualiton, as baryons in the hadron phase at low density. Unlike the low density phase, the parameters in the chiral Lagrangian can be calculated from the microscopic theory. For instance, the mass of Nambu-Goldstone bosons is found to be [24]

$$m_{NG}^2 \sim m_q^2 \Delta\bar{\Delta} \ln(\mu^2/\Delta^2)/\mu^2, \tag{48}$$

showing that mesons become massless at asymptotically large chemical potential, as the Dirac mass term, $m_q\bar{\psi}_+\psi_- \simeq (m_q^2/\mu)\psi_+^\dagger\psi_+$, vanishes for infinite density. (See also [25].) This is confirmed subsequently [26, 27]. Another interesting feature of meson mass is that the mass hierarchy is reversed [28]. For instance, if $m_s > m_{u,d}$,

$$m_K < m_\pi. \tag{49}$$

This inverse mass hierarchy is due to the fact that what we call a kaon in the CFL phase is the fluctuation of Cooper-pairs in the up and down flavor spaces,

$$U_{La\alpha}(x) \equiv \lim_{y\to x} \frac{|x-y|^{\gamma_m}}{\kappa} \epsilon^{ij}\epsilon_{abc}\epsilon_{\alpha\beta\gamma}\psi_{Li}^{b\beta}(-\vec{v}_F, x)\psi_{Lj}^{c\gamma}(\vec{v}_F, y), \tag{50}$$

where γ_m is the anomalous dimension.

6 Conclusion

I have discussed the exciting recent developments in color superconductivity in high density quark matter in terms of an effective theory formalism. I have shown that the effective theory calculation reproduces recent results on the Cooper pair gap, the critical temperature, and on the ground state of high density QCD. It not only simplifies the calculation very much but also allows us to estimate the critical density.

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